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LETTER TO THE EDITOR

Uniqueness of Hardy's state for fixed choice of observables

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Abstract. For a system of two spin-$\frac{1}{2}$ particles and for given four observables, two for each and non-commuting, there exists a unique state which admits Hardy's non-locality. Hence, no mixture state admits Hardy's non-locality.

Recently Hardy [1] has discovered that contradiction between local realism and quantum mechanics can be shown for a system of two spin-$\frac{1}{2}$ particles without using (Bell) inequalities. It has also been shown that this contradiction exists for almost all entangled states (maximally entangled states are exceptions) [2, 3]. Thomas F Jordan [4] has proved the converse of Hardy's result. He has shown that for any choice of two different measurement possibilities for each particle, a state can be found which gives Hardy-type contradiction. An easier proof has been provided by Mermin [5]. Jordan [6] has also shown that for a particular entangled state there are so many choices of observables which satisfy Hardy's non-locality condition.

Very recently Adan Cabello *et al* [6] proved the Bell–Kochen–Specker (BKS) theorem in four dimensions using 18 vectors only. There they gave a probabilistic proof of BKS theorem using only factorizable propositions, which can be interpreted in terms of local measurements and hence can be related to Hardy's non-locality theorem.

Using their technique we shall try to show that for two spin-$\frac{1}{2}$ particles and four spin-$\frac{1}{2}$ observables, two for each and non-commuting, there exists a unique state which satisfy Hardy's non-locality. So in the four-dimensional Hilbert space there cannot be any mixture which will show Hardy's non-locality. To prove this we shall use the following two premises,

(1) For a Hilbert space of dimension n, for given n orthogonal directions $\{\boldsymbol{r}_i\}$, one of them is labelled 1, $v(\boldsymbol{r}_j) = 1$ and the remaining ones 0, $v(\boldsymbol{r}_k) = 0$, $\boldsymbol{k} \neq j$.

(2) If for a subspace of dimension $\boldsymbol{k}(k < n)$, there are $\boldsymbol{k}$ number of linearly independent vectors for all of which $v(\boldsymbol{r}_i) = 0$, for $r_i = 1, 2 \ldots k$, then for any vector $\boldsymbol{r}$ in that subspace $v(\boldsymbol{r}) = 0$.

The first one is generally used for proving the BKS theorem. The simple reason that the second one holds, is the subspace of this kind is orthogonal to the state vector. So for any vector $\boldsymbol{r}$ in this subspace, $v(\boldsymbol{r}) = 0$.

We consider four propositions F and D for particle 1, and G and E for particle 2, where

$$F = \text{projection on vector } (1, 0)$$

$$D = \text{projection on vector } (d_1, d_2)$$

$$G = \text{projection on vector } (g_1, g_2)$$
$$E = \text{projection on vector } (e_1, e_2). \tag{1}$$

We want to find out the state which satisfies the following equations

$$\langle FG \rangle = 0$$
$$\langle D(1-G) \rangle = 0$$
$$\langle (1-F)E \rangle = 0$$
$$\langle DE \rangle > 0. \tag{2}$$

The above equations of (2) can be translated into answers to the projectors over the following four-dimensional vectors

$$\boldsymbol{v}(g_1, g_2, 0, 0) = 0 \tag{3}$$
$$\boldsymbol{v}(d_1g_2, -d_1g_1, d_2g_2, -d_2g_1) = 0 \tag{4}$$
$$\boldsymbol{v}(0, 0, e_1, e_2) = 0. \tag{5}$$

There is a non-zero probability for

$$\boldsymbol{v}(d_1e_1, d_1e_2, d_2e_1, -d_2e_2) = 1. \tag{6}$$

It is to be noted that four vectors in (3)–(6) are linearly independent. So they generate the four-dimensional Hilbert space associated with the system. Again the vector in (3) is orthogonal to the vectors (4) and (5). Now we consider another vector $\boldsymbol{k}$ (say) which is orthogonal to the vector in (4) and lie in the plane generated by vectors in (4) and (5). According to the second premise the value assigned to this new vector must be zero, i.e.

$$\boldsymbol{v}(\boldsymbol{k}) = 0. \tag{7}$$

Let $\boldsymbol{m}$ be the vector which is orthogonal to the subspace generated by the vectors in (3), (4) and $\boldsymbol{k}$. Then from the second condition we get

$$\boldsymbol{v}(\boldsymbol{m}) = 1. \tag{8}$$

This vector $\boldsymbol{m}$ represents the state which is unique. The fourth condition is satisfied because vector in (6) is not in the subspace generated by vectors in (3), (4) and $\boldsymbol{k}$. This completes the proof.

The vector $\boldsymbol{m}$ can be easily found from the orthogonality of $\boldsymbol{m}$ to the vectors in (3), (4) and (5). If we write

$$\boldsymbol{m} = (e, f, g, h) \tag{9}$$

then

$$e = d_2g_2(e_1g_1 + e_2g_2)$$
$$f = -d_2g_1(e_1g_1 + e_2g_2)$$
$$g = -e_2d_1(g_1^2 + g_2^2)$$
$$h = e_1d_1(g_1^2 + g_2^2). \tag{10}$$

It can be easily shown that the vector $\boldsymbol{m}$ cannot represent vectors for product state or maximally entangled state.

The scalar product between $\boldsymbol{m}$ and vector in (6) is

$$S = d_1d_2(e_1g_1 + e_2g_2)(e_1g_2 - e_2g_1). \tag{11}$$

None of the factors in the above expression can be zero, because then projectors for the same particle would commute.

It is a well known result that there are mixture states for two spin-$\frac{1}{2}$ particle system which violate the Bell-CHSH inequality [7]. But our result shows that unlike in the case of Bell-CHSH inequalities, no mixture state for two spin-$\frac{1}{2}$ particles admit non-locality without inequalities.

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